The Second-Law Based Thermodynamic Optimization Criteria for Pulse Tube Refrigerators

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ABSTRACT

The second-law based thermodynamic optimization criteria with application to Stirling and pulse tube cryogenic refrigerators are investigated and compared. New ecological criteria for optimization and characterization of cryogenic refrigerators are proposed and a model based on the exergy flow interpretation of the criteria is developed. It is shown that the conventional ecological criteria used in the analysis and optimization of energy systems is not suited for application to cryogenic refrigerators. The criteria must be modified due to the fact that practical cryogenic refrigerators have high internal irreversibilities. This high internal irreversibility exists despite the fact that the external irreversibility due to heat transfer is quite low in cryogenic refrigerators. It is shown how the proposed ecological criteria simulate the condition where a compromise between exergy delivered and exergetic efficiency in design space is obtained. The optimization criteria is applied to experimental data generated for a high efficiency cooler by assuming a linear load curve for the cooler while appropriate parameter models are chosen to simulate other cooler characteristics. Numerical example for the effect of different parameters including the heat rejection temperature, no load temperature, and other important system parameters are presented. Finally, application of the criteria to thermodynamic optimization of multistage cryocoolers is discussed.

INTRODUCTION

The two most important quantities characterizing thermodynamic performance of an energy system are the power and efficiency of the system. The effect of system parameters on the power and efficiency can be conveniently represented by a power/efficiency diagram. The power is normally defined as the rate of production of the energy of interest and efficiency is the rate of energy production per unit of input energy. Depending on the constraints on the energy system a compromise between the power and efficiency may occur.[1] This compromise between power and efficiency has been the subject of a debate among several investigators.[2,3] In the application to cryo-

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genic refrigerators the intendency exists to discuss the effect of the constraints on the power/efficiency diagram using examples from pulse tube refrigerators (PTRs). In application to refrigerators and cryocoolers, the product of the energy system is the cooling capacity and efficiency is normally represented by the Coefficient of Performance (COP). A rational basis for defining the efficiency is the second-law efficiency, which is a comparison of the COP of the system with the COP of a totally reversible (Carnot) system. This is equivalent to considering the cooling power to be represented by the rate of exergy of the product delivered to the cold reservoir and the input power as the rate of exergy input to the system at the compressor.[4,5] The function of PTRs, that are used as an example in this study, is to transfer the exergy of input in the form of compressor power to output exergy to the cold reservoir. When a compromise between power and efficiency exists the energy system can be optimized based on an ecological objective function. The ecological criterion for optimization of energy systems, has been previously applied to the optimization of refrigeration systems.[6,7] Due to high internal irreversibility of the cryogenic refrigerators, the criterion must be modified for use with PTRs. A modified ecological criterion is proposed for use with PTRs and a simple analytical model is developed to show its application.

THERMODYNAMIC MODEL

In order to quantify and discuss the compromise between cooling capacity and the efficiency, a simple analytical model of the regenerator and energy balance of the PTR is considered. Assuming the pulse tube to be ideal, the timed-average enthalpy flow at the cold side of the pulse tube over a cycle can be written as[8]

$$\dot{H}_2 = (1/2)\frac{p_2}{P_a}\dot{m}_2 T_c R Cos(\phi_2 - \theta_2) \tag{1}$$

where $\dot{m}_2$ is the amplitude of mass flow rate, p_2 is the amplitude of pressure, $(\phi_2 - \theta_2)$ is the phase angle between the mass flow and pressure, R is the gas constant, and T_c is the cold end temperature, P_a the average pressure, and subscript 2 refers to the cold side of pulse tube. The conservation of energy on the Cold Heat Exchanger (CHX) can be written as

$$\dot{Q}_c = \dot{H}_2 - \dot{Q}_{reg} - \dot{Q}_{cond} \tag{2}$$

where $\dot{Q}_c$ is the cooling load, $\dot{Q}_{reg}$ is the energy transfer from the regenerator due to ineffectiveness of the regenerator, and $\dot{Q}_{cond}$ is the heat transfer to the CHX by conduction. $\dot{Q}_{reg}$ can be estimated using the ineffectiveness of the regenerator,

$$\lambda = \frac{\dot{Q}_{reg}}{\dot{H}_1^+ - \dot{H}_2^-} = \frac{\dot{Q}_{reg}}{(1/\pi)C_p(T_o\dot{m}_1 - T_c\dot{m}_2)} \tag{3}$$

where the denominator represents the maximum rate of enthalpy transfer into the regenerator and the subscript 1 refers to the hot side of the regenerator. Conduction heat transfer to the cold heat exchanger can be estimated assuming a linear temperature profile in the regenerator,

$$\dot{Q}_{cond} = (KA/L)(T_o - T_c) \tag{4}$$

where KA/L is effective thermal conductance in the regenerator and its shell and T_o is the temperature of the environment.

To simplify the analysis we assume that the phase shifter can be controlled such that the mass flow rate and pressure are in phase at the cold side of the pulse tube. Using the above equations the cooling load can be written as

$$\dot{Q}_c = (1/2)RT_o\dot{m}_2\frac{p_2}{P_a} - (1/\pi)\lambda C_p(T_o\dot{m}_1 - T_c\dot{m}_2) - (KA/L)(T_o - T_c) \tag{5}$$

Heat and mass transfer in regenerators is very complex. For thermodynamic analysis we consider a simple model for the regenerator to find the bounds for cooling capacity and efficiency of PTRs. It is assumed that the mass flow rate at the cold side of the regenerator can be obtained using a linear relation based on a given appropriately average flow conductance.

$$\dot{m}_2 Cos(\omega t - \phi_2) = C\left[p_1 Cos(\omega t - \theta_1) - p_2 Cos(\omega t - \theta_2)\right] \quad (6)$$

where C is average flow conductance in regenerator. In addition, a simple model for the conservation of mass in the regenerator can be written as,

$$\dot{m}_1 Cos(\omega t - \phi_1) = \dot{m}_2 Cos(\omega t - \phi_2) - V\omega\left[p_1 Sin(\omega t - \theta_1) + p_2 Sin(\omega t - \theta_2)\right] \quad (7)$$

where V is the properly averaged effective regenerator void volume including the influence of temperature distribution in the regenerator and ω is the angular velocity. Expansion of Eqs. (6) and (7), and applying the trigonometric identities, results in a set of equations that uniquely determine the relations between the phase shifts and amplitudes of pressure and mass flow rates at the cold and hot sides of the regenerator. Using Eqs. (5), (6) and (7), the normalized cooling capacity $\dot{Q}_{cn}=2P_a\dot{Q}_c/CRT_0\, p_1^2$ can be written as

$$\dot{Q}_{cn} = \frac{2P_a\dot{Q}_c}{CRT_o p_1^2} = (1-\Pr)\left[\Pr T_c/T_o - \beta(\sqrt{1+\alpha^2(1+\Pr)^2/(1-\Pr)^2} - T_c/T_o)\right] \quad (8)$$

where $\Pr = p_2/p_1$ is the pressure amplitude ratio across regenerator, and the dimensionless parameter α and β are given by

$$\alpha = V\omega/C$$
$$\beta = 2\lambda P_a\gamma/\pi p_1(\gamma-1) \quad (9)$$

In Eq. (9) γ is the specific heat ratio of the working fluid**.** Therefore, the dimensionless cooling load is a function of four dimensionless parameters α, β, Pr and T_c/T_o. The magnitude of exergy transfer to the cold reservoir is defined by[4]

$$\dot{E}_c = \dot{Q}_c(T_o/T_c - 1) \quad (10)$$

Using Eqs. (8) and (10), the normalized exergy transfer to the cold reservoir can be written as

$$\dot{E}_{cn} = (1-\Pr)\left[\Pr T_c/T_o - \beta(\sqrt{1+\alpha^2(1+\Pr)^2/(1-\Pr)^2} - T_c/T_o)\right](T_o/T_c - 1) \quad (11)$$

where $\dot{E}_{cn}=2P_a\dot{E}_c/CRT_0\, p_1^2$. Considering an ideal compressor, the power input at the compressor side can be written as

$$\dot{W} = (1/2)\frac{p_1}{P_a}\dot{m}_1 T_o R Cos(\phi_1 - \theta_1) \quad (12)$$

where subscript 1 refers to the parameters at the compressor side (hot side of regenerator). From Eqs. (11) and (12), the exergetic efficiency can be obtained.

$$\eta_{ex} = \left[\Pr T_c/T_o - \beta(\sqrt{1+\alpha^2(1+\Pr)^2/(1-\Pr)^2} - T_c/T_o)\right](T_o/T_c - 1) \quad (13)$$

ECOLOGICAL OPTIMIZATION CRITERIA

From Eqs. (8) or (11) and (13) it can be seen that, depending on the constraint on the system, the optimum cooling power and efficiency can occur at two different points. It is a well established fact that, for practical reasons, actual systems operate under conditions between these two optima. This in general can be achieved using a profit function based on the economic consideration. For application to refrigeration cycles, based on the viewpoint of exergy analysis, a unified ecological optimization criterion has been proposed for generalized irreversible Carnot refrigerators.[6] Because of high irreversibility of cryocoolers, conventional ecological objective functions are not suitable for application to PTRs. We propose the following objective function as the best compromised between exergy output of the cryocooler and its exergy destruction (irreversibility) as,

$$\dot{E}_{eco} = \dot{E}_c - \eta_{ex}\dot{E}_d \quad (14)$$

where η_{ex} is exergetic efficiency (second-law efficiency) and subscripts eco, c, and d represent ecological, cold reservoir, and destruction, respectively. Equation (14) can also be written in terms of the ecological objective function for energy denoted by ECO,

$$ECO = \dot{Q}_c - COP(T_0\,\dot{S}_{gen}) \tag{15}$$

where COP is the Coefficent of Performance for the cryocooler and $\dot{S}_{gen}$ is the rate of entropy generation for the cryocooler. Using the exergy balance for the PTRs and Eqs. (11), (13), and (14), the normalized ecological objective function for PTRs can be written as

$$\dot{E}_{eco,n} = (1-\mathrm{Pr})\left[(\mathrm{Pr}\,T_c/T_o - \beta(\sqrt{1+\alpha^2(1+\mathrm{Pr})^2/(1-\mathrm{Pr})^2} - T_c/T_o))(T_o/T_c - 1)\right]^2 \tag{16}$$

where $\dot{E}_{eco,n}=2P_a\dot{E}_{eco}/\dot{C}RT_0\,p_1^2$.

The load curve is one of the most basic pieces of information that is reported for the performance evaluation of the cryocoolers. The load curve is usually a nearly linear variation of rate of the heat transfer from the load as a function of the cold end temperature. This is consistent with the enthalpy flow model, and many experimental results as well.[8,9] Therefore, the load curve in general can be represented with a reasonable accuracy as a linear function given by,

$$\dot{Q}_c = s(\dot{W},T_o)\left(T_c - T_{co}(\dot{W},T_o)\right) \tag{17}$$

where s is the slope of the load curve and T_{co} is the no load temperature of the cryocooler. In general, both of these parameters are functions of several variables. In this study we assume that they are functions of two basic parameters; the heat rejection temperature T_0 and the input power. Using Eqs. (10) and (17), it is easy to show that for constant input power, the magnitude of exergy transfer to the cold reservoir is maximum at the geometric mean of the heat rejection temperature and no load temperatures and is independent of the slope of the load curve.[5]

$$T_{c,opt} = \sqrt{T_o T_{co}} \tag{18}$$

Based on theoretical consideration, and characterization curves of TRW 95 K High Efficiency Cryocooler (HEC) in our laboratory, $s(\dot{W}, T_0)$ and $Tco(\dot{W}, T_0)$ can be fitted parametrically with reasonable accuracy by the following functions.

$$\begin{aligned} T_{co}/T_o &= (1+a\dot{W}^b)^{-1} \\ sT_o &= \dot{W}(1+1/a\dot{W}^b)(1+c\dot{W}^d)^{-1} \end{aligned} \tag{19}$$

where *a*,*b*,*c*,and *d* are parameters that can be obtained from fitting the equations to the results of experimental data.

RESULTS AND DISCUSSIONS

Equations (11), (13), and (16) show that $\dot{E}_{cn}$, η_{ex}, and $\dot{E}_{eco,n}$ are functions of dimensionless variables α, β, T_0/T_c, and Pr. Therefore, these quantities can be easily optimized with respect to Pr if the other parameters are held constant. It can easily be seen from the equations that the optimum Pr is different for the three quantities.

Figure 1 shows the variation of the exergy delivered to the cold reservoir, the exergetic efficiency, and the ecological objective function as a function of the pressure amplitude ratio for the values of $\alpha = 0.2$, $\beta = 0.1$, and $T_0/T_c = 3.75$. The numerical values of the optima for the three quantities are also given in the same figure for comparison. The values of ecological function are considerably smaller than the exergy transfer to the cold reservoir due to the large irreversibility of typical PTRs.

When the parameters α, β, and Pr are fixed and load temperature is a variable, it can be shown that the input power is fixed in the example problem under investigation. For this case the optimum load temperature for the three second-law based objective functions are the same and is given by Eq. (18).

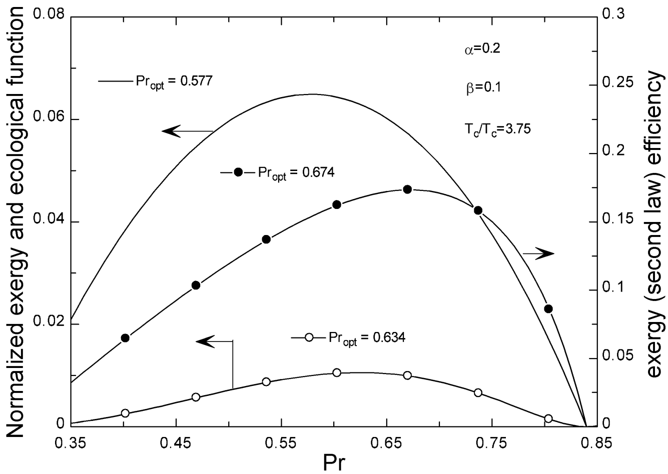

Figure 1. Normalized exergy, exergetic efficiency and ecological function as a function of amplitude pressure ratio across regenerator.

$$T_{c,opt} = \sqrt{T_o T_{co}} = T_o \sqrt{\beta \sqrt{1+\alpha^2 (1+\mathrm{Pr})^2 / (1-\mathrm{Pr})^2} \,/(\mathrm{Pr}+\beta)} \tag{20}$$

For example, for the values of $\alpha = 0.2$, $\beta = 0.1$, Pr = 0.6, and $T_0 = 300$ K, the optimum cooling load temperature would be $T_{c,opt} = 128.32$ K.

Figure 2 shows the power/efficiency diagram for $\alpha = 0.2$, $\beta = 0.1$, and three values *of* T_0/T_c of 3, 3.75, and 5. In this figure the pressure amplitude ratio, Pr, is a variable for each curve. The looped-shape curves are indicative of a compromise between power and efficiency. In this case the optimum cooling power and optimum efficiency correspond to two different values of Pr and design conditions. Therefore, the cooler can be optimized for maximum efficiency, maximum cooling power, or another objective function for a compromise between the two. The ecological objective function offers a method for optimizing the cooler for a compromise between maximum cooling power and maximum efficiency (minimum irreversibility). For example, in Fig. 1 for $T_0/T_c = 3.75$ the optimum ecological function corresponds to Pr = 0.624. Comparing the conditions of the optimum exergy transfer to the cold reservoir and the optimum ecological objective function, the exergy

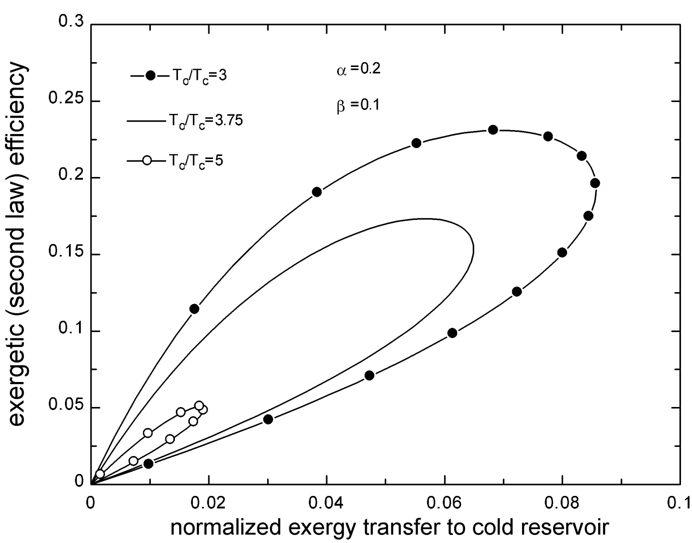

Figure 2. Power efficiency diagram with pressure amplitude ratio across regenerator as a parameter.

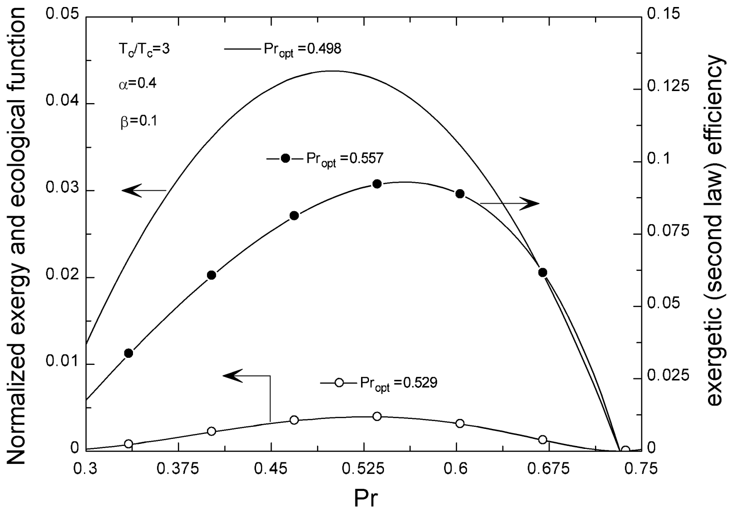

Figure 3. Normalized exergy, exergetic efficiency and ecological function as a function of amplitude pressure ratio across regenerator.

transfer at the optimum ecological objective function is smaller than the optimum exergy transfer to the cold reservoir by 3.2 percent. At the same time the efficiency corresponding to the optimal ecological criteria is higher than the efficiency at the optimum exergy transfer rate by 9.9 percent. It is interesting to note that the comparison of the two optima results in a reduction of 10.7 percent in the value of the irreversibility when the system is optimized based on the ecological criterion as compared to when it is optimized based on exergy transfer criteria.

Figure 3 represents the same results as Fig. 1 for the values of $\alpha = 0.4$, $\beta = 0.1$, and $T_0/T_c = 3$. In this example where the angular velocity and flow conductance are fixed, the change can be accomplished by controlling the flow area and the hydraulic diameter in the regenerator. An increase in the value of parameter α in this case corresponds to the condition of a higher void volume in the regenerator. The comparison of Figs. 1 and 3 clearly shows that while the optimum values for Pr does not change very much, the degradation in the cooler performance in both power and efficiency is considerable when the value of is increased from 0.2 to 0.4. This is even true considering that smaller value of T_o/T_c is used in Fig. 3.

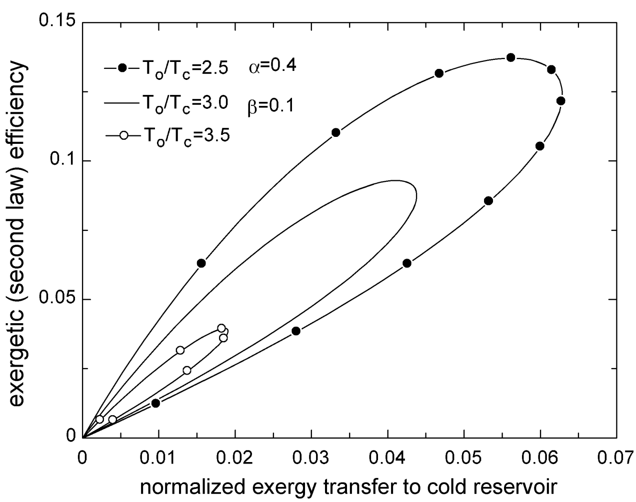

Figure 4. Power efficiency diagram with pressure amplitude ratio across regenerator as a parameter.

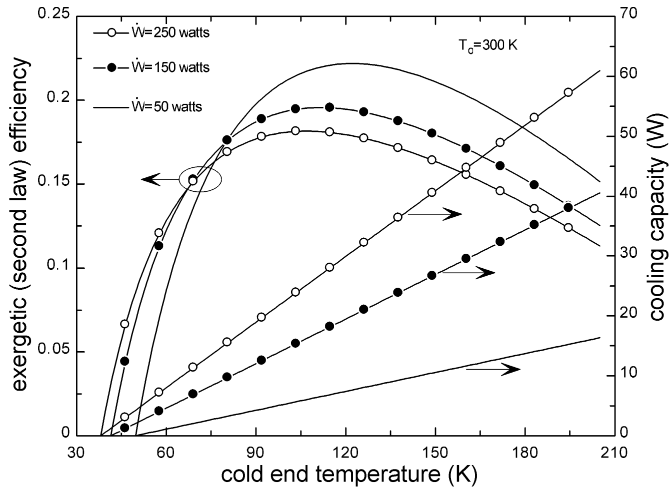

Figure 5. Cooling capacity and efficiency for HEC cooler as a function of the cold end temperature.

Figure 4 shows similar results to Fig. 2 for the values of $\alpha = 0.4$, $\beta = 0.1$, and three values T_0/T_c of 2.5, 3, and 3.5. The looped shape curve corresponding to T_0/T_c = 3.5 has a very small area indicating the proximity of the operating condition to the no load temperature for this example. Comparison of Figs. 2 and 4 shows the significance of the dimensionless parameter α on cooling power and efficiency diagram when β is fixed. Both Figs. 2 and 4 show that as the no load condition is approached; the distinction between the optimum cooling power and optimum efficiency is diminished. A value of T_0/T_c = 3.5 is used in Fig. 4; the results for the value of T_0/T_c = 3.5 is very close to the no load temperature for this example.

Figure 5 shows the exergetic efficiency and the cooling capacity as a function of the cold end temperature for the heat rejection temperature of T_o=300 K for three different values of input power. For a fixed input power the cooling capacity increases linearly with the cold end temperature as Eq. (17) indicates. The exergetic efficiency has a maximum at the temperature given by Eq. (18). The results presented in Fig. 5 is based on the curve fit of Eq. (19) to the results of the load curve for TRW 95 K HEC cooler at the heat rejection temperature of T_o=300 K.[5] The power/efficiency diagram corresponding to the parametric representation given by Eq. (18) does not produce a loop-shaped curve under the constraint of constant power with the cold end temperature as a parameter. In fact, a loop-shaped curve is observed for the HEC cooler when the effect of the cold end temperature on input power is considered.[5] The fact that the rate of exergy transfer to the cold reservoir has an optimum has application in the design of multistage PTRs and the optimum selection of the temperatures of intermediate stages.

CONCLUSIONS

A second-law based ecological criterion is proposed as an objective function for optimization of cryogenic refrigerators. Assuming a controlled phase shifter that can produce a condition where the mass flow rate and pressure are in phase at the cold side of pulse tube, a model is developed to show the application of the ecological optimization criteria for PTRs. It is shown that loop-shaped curves, indicating a compromise between cooling power and efficiency, can be obtained for PTRs depending on the constraints of the system. In general, the existence of a compromise between cooling power and efficiency depends on the definition of these quantities and the constraints on the system. The results are given in dimensionless quantities suitable for parametric studies. The thermodynamic bound for performance of PTRs can be conveniently obtained in terms of the dimensionless numbers developed in this study.

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