Optimum Cooling Capacity of Two-Stage Thermoelectric Refrigerators Operating at Cryogenic Temperatures

A. Razani[1], **C. Dodson**[2], **T. Fraser**[2]

[1]The University of New Mexico
Albuquerque, NM 87131

[2]Spacecraft Component Thermal Research Group
Kirtland AFB, NM 87117-5776

ABSTRACT

Optimization of two-stage thermoelectric refrigerators is considered in this study. Cooling capacity of the refrigerator under a cascade arrangement having different electric currents and numbers of thermocouples at each stage is optimized using the method of Lagrange multipliers. Emphasis is placed on the effect of thermal conductance at the hot side on optimum cooling capacity and the resulting cooling flux at the cold side of the refrigerator. The effects of electric contact resistance, the figure of merit of the thermoelectric material, and the length of the elements at each stage are considered in the analysis. The performance of the refrigerator operating at low temperatures while rejecting heat to a reservoir at cryogenic temperature is evaluated. Important non-dimensional variables convenient for parametric studies and design analysis are developed and their effect on the optimum performance of the refrigerator is presented.

INTRODUCTION

Thermoelectric refrigerators possess great advantages for cooling of space-based infrared detectors because they are solid state devices having no moving parts and are miniature, highly reliable, and easy to integrate into the system. They provide the advantage of high cooling fluxes and spot cooling capabilities compared to conventional cryogenic refrigerators.[1,2] The development of thermoelectric refrigerators for application at cryogenic temperatures is hampered by the fact that thermoelectric materials for application at low temperatures are not available and challenges exist in improving their cooling capacity and efficiency.[3-4] New efforts are underway to develop thermoelectric materials with application to cryogenic temperatures.[5] In this study we consider a two-stage thermoelectric refrigerator and investigate its optimization concentrating on the most important internal losses and the effect of thermal conductance for heat rejection at the hot side of the refrigerator. We optimize the cooling capacity of a two-stage cascade thermoelectric refrigerator with a separate electric current used at each stage. The

Cryocoolers 18, edited by S.D. Miller and R.G. Ross, Jr.

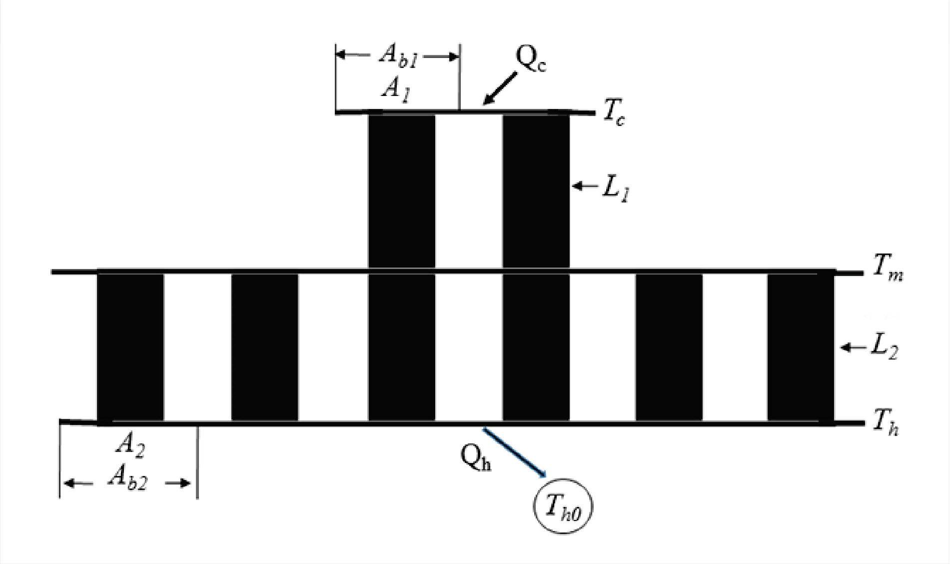

Figure 1. The schematic of a two-stage thermoelectric refrigerator with the basic parameters used in the model.

optimization of two-stage thermoelectric refrigerator, using different methods and arrangements, have been subject of several studies.[6-10] In this study we are concerned with optimizing the cooling capacity of a two-stage thermoelectric refrigerator in terms of the most important non-dimensional parameters including the most important losses in the refrigerator. We are also interested in the cooling flux that can be provided by the two-stage refrigerator.[11] We discuss the effect of important normalized parameters on the performance of the refrigerator operating at cryogenic temperatures.

MODELING AND OPTIMIZATION OF THERMOELECTRIC REFRIGERATORS

Figure 1 shows the schematic of a two-stage thermoelectric refrigerator and the important parameters used in the model. We assume that a thermal reservoir at the hot side is available or is supported by another cooler that operates at cryogenic temperatures. In this manner, a thermoelectric refrigerator can be considered to be operating at cryogenic temperatures. In addition, the effect of important system parameters on the performance of the refrigerator can be assessed. The number of thermoelectric elements at each stage as well as the area aspect ratio of each thermoelectric element and their lengths can be different and are used as control parameters. The thermal resistance between the two stages is neglected and the thermal spreading resistance at the hot side is included in the thermal conductance for heat rejection to the auxiliary cooler or a high temperature reservoir. Electric resistance is modified to include the contact resistance defining an effective electric resistance for the elements at each stage. Heat leak to the thermoelectric element has been neglected in this study. Assuming average electrical and thermal properties at each stage we evaluate the effect of the properties on the optimum cooling capacity of a two-stage thermoelectric refrigerator. We consider a cascade arrangement of the two stage thermoelectric cooler with independent electric current to each stage.[10] Another goal of this study is to develop the important normalized parameters affecting the optimum performance of the two-stage thermoelectric refrigerator. The cooling capacity of the two-stage thermoelectric refrigerator can in general be considered a function of the electric current at the first stage, the temperature of the cold stage, the temperature of the mid-stage, and the thermoelectric properties of the first stage. The normalized cooling capacity of the two-stage thermoelectric refrigerator can be written as[12]

$$Q_{cn} = I_{1n}T_{cn} - 0.5I_{1n}^2 / Z_{eff1}T_{ho}A_{1n} - A_{1n}(T_{mn} - T_{cn}) \tag{1}$$

where the normalized parameters are defined as follows,

$Q_{cn} = Q_c L_1/2N_1 A_{b1} k_1 T_{ho}$, normalized cooling capacity (2)
$I_{1n} = \alpha_1 I_1 L_1/A_{b1} k_1$, normalized electric current to the first stage (3)
$A_{1n} = A_1/A_{b1}$, normalized area of TE element of the first stage (4)
$T_{cn} = T_c/T_{ho}$, normalized temperature of the cold side (5)
$Z_{eff1} = \alpha_1^2/[k_1(\rho_1 + 2\rho_{c1}/L_1)]$, effective figure of merit for the first stage (6)
$T_{mn} = T_m/T_{ho}$, normalized inter-stage temperature (7)

Other parameters are shown in the list of nomenclature. Assuming no heat leakage to the inter-stage, the continuity of energy at the mid-stage can be written as,

$$\begin{aligned}&[I_{1n}T_{mn} + \frac{0.5I_{1n}^2}{Z_{eff1}T_{ho}A_{1n}} - A_{1n}(T_{mn} - T_{cn})] - \ [I_{2n}T_{mn} - \frac{0.5I_{2n}^2}{Z_{eff2}T_{ho}A_{2n}} \\ &\quad -A_{2n}(T_{hn} - T_{mn})]\chi{=}0\end{aligned} \tag{8}$$

where the normalized variables are defined as,

$\chi = N_2 A_{b2} k_2 L_1/N_1 A_{b1} k_1 L_2$, normalized aspect ratio of the two stages (9)
$I_{2n} = \alpha_2 I_2 L_2/A_{b2} k_2$, normalized electric current to the second stage (10)
$A_{2n} = A_2/A_{b2}$, normalized area of TE element of the second stage (11)
$T_{hn} = T_h/T_{ho}$, normalized temperature of the hot side (12)
$Z_{eff2} = \alpha_2^2/[k_2(\rho_2 + 2\rho_{c2}/L_2)]$, effective figure of merit for the second stage (13)

The energy balance at the hot side of the refrigerator in terms of normalized parameters can be written as,

$$I_{2n}T_{hn} + \frac{0.5I_{2n}^2}{Z_{eff2}T_{ho}A_{2n}} - A_{2n}(T_{hn} - T_{mn}) - U_{hn}(T_{hn} - 1) = 0, \tag{14}$$

where,

$U_{hn} = U_h L_2/k_2$, the normalized thermal conductance at the hot side (15)

It should be pointed out that the normalized thermal conductance given by Eq. (15) includes the effect of thermal spreading resistance at the hot side of the thermoelectric cooler. One of the goals of this study is to investigate the effect of the thermal conductance at the hot side on the optimum performance of two-stage thermoelectric refrigerators operating at the cryogenic temperatures.

The normalized cooling capacity given by Eq. (1) can be optimized with Eqs. (8) and (14) as the constraints using the method of Lagrange Multipliers.[12,13] These equations can be written as,

$$T_{cn} - \frac{I_{1n}}{Z_{eff1}T_{ho}A_{1n}} - \ \lambda_2\left(T_{mn} + \frac{I_{1n}}{Z_{eff1}T_{ho}A_{1n}}\right) = 0, \tag{16}$$

$$-\lambda_1(T_{hn} - \frac{I_{2n}}{Z_{eff2}T_{ho}A_{2n}}) + \ \lambda_2\left(T_{mn} - \frac{I_{2n}}{Z_{eff2}T_{ho}A_{2n}}\right)\chi = 0, \tag{17}$$

$$\lambda_1(T_{hn} - A_{2n} - U_{hn}) + \ \lambda_2 A_{2n}\chi = 0, \tag{18}$$

$$A_{1n} + \lambda_1 A_{2n} + \lambda_2[I_{1n} - A_{1n} - (I_{2n} + A_{2n})\chi] = 0. \tag{19}$$

Equations (16) to (19) together with the constraints given by the Equations (8) and (14) are six Equations and six unknowns for the normalized electric currents at the first and second stages I_{1n}, I_{2n}, the normalized temperatures at the hot side and inter-stage T_{hn} and T_{mn} and the Lagrange multipliers L1 and L2. The values for of A_{1n}, A_{2n}, T_{cn}, Z_{eff1}, Z_{eff2}, T_{ho}, U_{hn}, and χ are considered to be the control parameters. The normalized aspect ratio of the two stages, χ, plays an important role in the performance and design of two-stage thermoelectric refrigerators as has been discussed previously.[10] This parameter represents the comparison of cooling capacity of the first stage to the cooling capacity of the second stage. For the case of the same average thermal conductivity of the two stages Eq. (9) shows that the normalized parameter χ reduces to a geometry factor and the number of thermoelectric elements used in the two stages.

When the thermal conductance at the hot side of the thermoelectric refrigerator approaches infinity, Eqs. (8), (14), and (16) to (19) reduce to four equations and four unknowns for I_{1n}, I_{2n}, λ_2 and and T_{mn}. In this case $\lambda_1 \to 0$ as $U_{hn} \to \infty$ and $T_{hn} \to 1$. This ideal case has been used to check the solution of the six nonlinear equations mentioned previously for large values of thermal conductance at the hot side of the refrigerator. Equations (17) and (19) are nonlinear equations for the case of no thermal resistance at the hot side of the refrigerator and can be simplified and written as,

$$T_{mn} = I_{2n}/Z_{eff2}T_{h0}A_{2n} \text{ ,} \tag{20}$$

$$\lambda_2 = -A_{1n}/[I_{1n} - A_{1n} - (I_{2n} + A_{2n})\chi]. \tag{21}$$

Equations (20) and (21) together with Eqs. (8) and (16) constitute four nonlinear equations and four unknowns for T_{mn}, λ_2, I_{1n}, I_{2n}, when $U_{hn} \to \infty$ and $T_{hn} \to 1$. Furthermore, these equations can be analytically converted into two nonlinear equations and two unknowns for I_{1n} and I_{2n} to find the normalized electric currents for the optimum cooling capacity when no thermal boundary resistance exists at the hot side of two-stage thermoelectric refrigerators.

RESULTS AND DISCUSSION

Cooling capacity and the coefficient of performance (COP) are the most important quantities for the performance evaluation of refrigerators. Thermal boundary resistances at the hot and cold sides of TERs are mainly due to the connection of the hot and cold sides to the reservoirs. It is assumed that the thermal resistance at the cold side of the refrigerator is negligible. Therefore the temperatures of the cold side of the refrigerator and the cold reservoir are the same. In addition the effect of heat leak to the refrigerators is neglected. The thermal resistance at the hot side of thermoelectric refrigerator is especially important for space applications where the heat must be rejected through the radiator at the hot side of the refrigerator. The thermal resistances can also be due to thermal contact resistance and thermal spreading resistance. In this study we use the normalized thermal conductance parameters at the hot side to represent the thermal boundary resistances between the hot reservoir and the thermoelectric refrigerator. We concentrate on the effect of the most important systems parameters on the optimum cooling capacity of the two-stage thermoelectric refrigerator. The temperature of the hot reservoir is fixed at 100 K. The figure of merit is assumed to be fixed at *Zeff*=0.005 in most calculations simulating the thermoelectric properties of $Bi_{1-x}Sb_x$ with x = 16.5 percent reported for a potential thermoelectric application at the cryogenic temperatures.[3,4] The figure of merit is calculated using nominal values reported in the literature for the thermal conductivity $k = 2.4$ W/mK, the electric resistivity $\rho = 1.2\times10^{-6}$ Ωm, and the Seebeck coefficient $\alpha = 0.00012$ V/K for $Bi_{0.835}Sb_{0.165}$ thermoelectric material. It should be pointed out that the effective figure of merit for the thermoelectric materials, defined in this study, includes the electric contact resistance at the two stages as given by Eqs. (6) and (13).

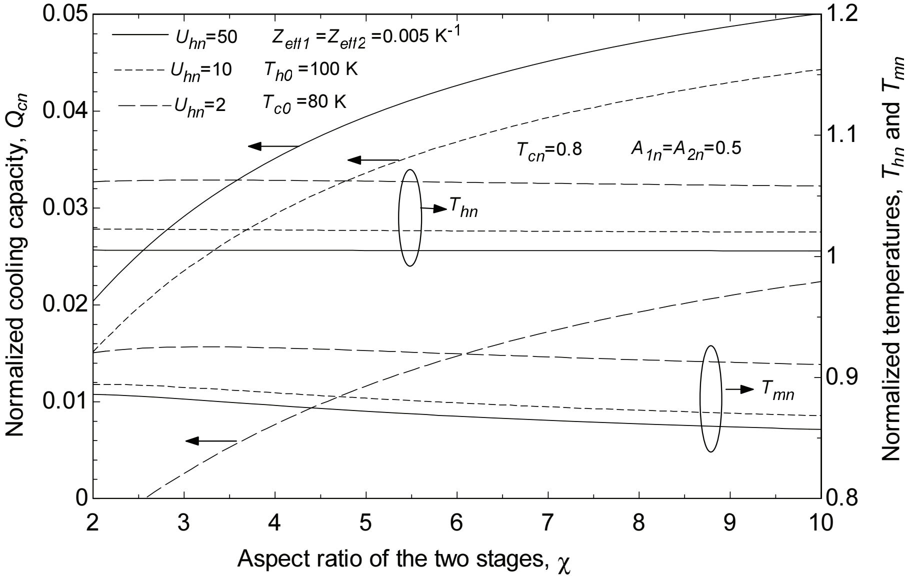

Figure 2. The normalized optimum cooling capacity and the normalized hot and inter-stage temperatures as a function of aspect ratio of the two stages for three values of normalized thermal conductance at the hot side.

In most calculations we use the same effective figure of merit for both stages but the formulation is general and different values can be used in parametric studies.

Figure 2 shows the normalized maximum cooling capacity on the left Y-axis as a function of the aspect ratio of the two stages defined by Eq. (9) for three values of normalized thermal

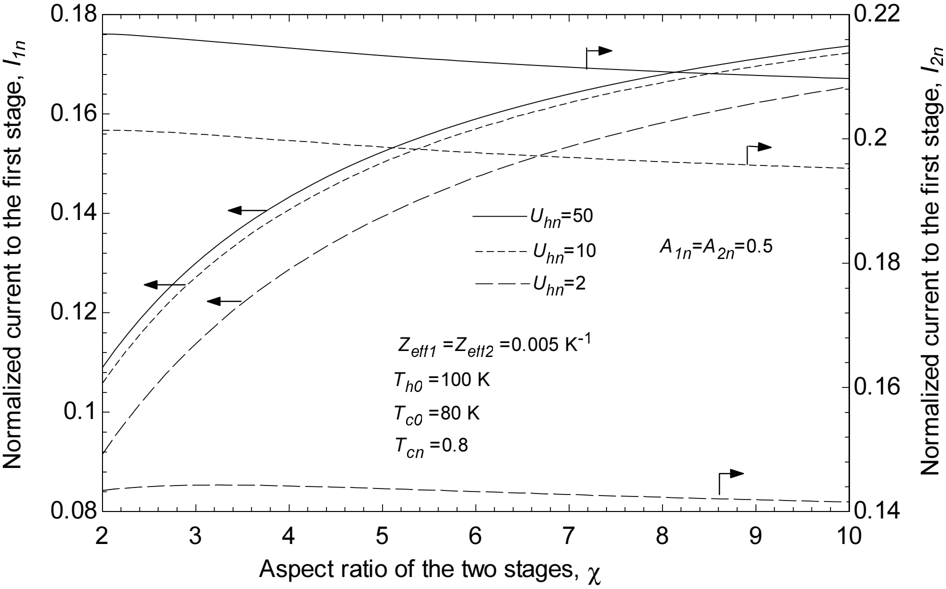

Figure 3. The normalized electric current to the first and second stages as a function aspect ratio of the two stages for the three values of thermal conductance at the hot side.

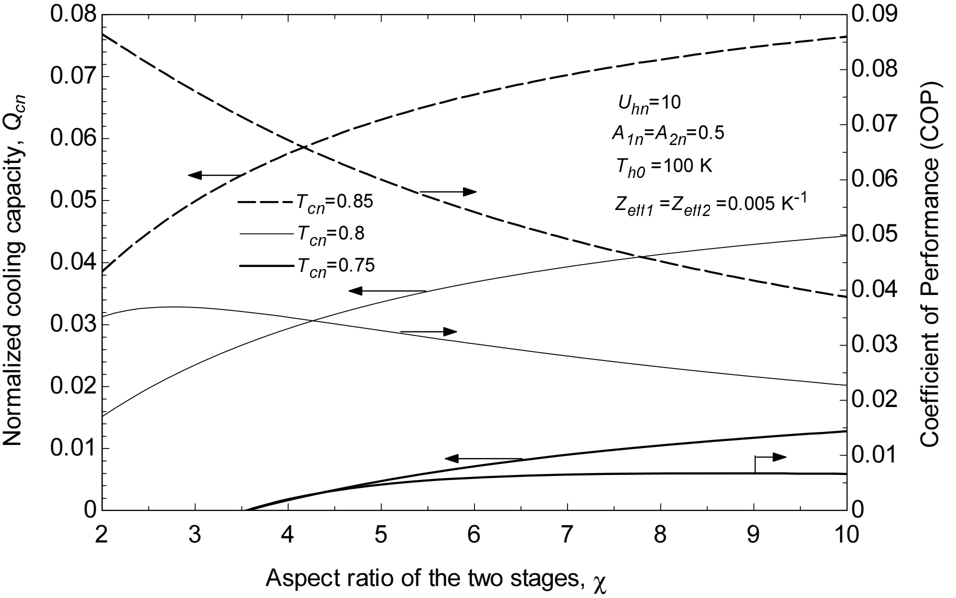

Figure 4. The normalized optimum cooling capacity and corresponding COP as a function of aspect ratio of the two stages for three values of normalized cold end temperature.

conductance at the hot side of the refrigerator. The normalized temperatures of the hot side and the mid-stage are given on the right Y-axis for the same condition. Other parameters of interest used in the calculations are indicated on the figure. The value of $U_{hn} = 50$ is considered to be high indicating a very small thermal resistance at the hot side of the refrigerator. This can be easily seen by comparing the temperature at the hot side of the refrigerator approaching the

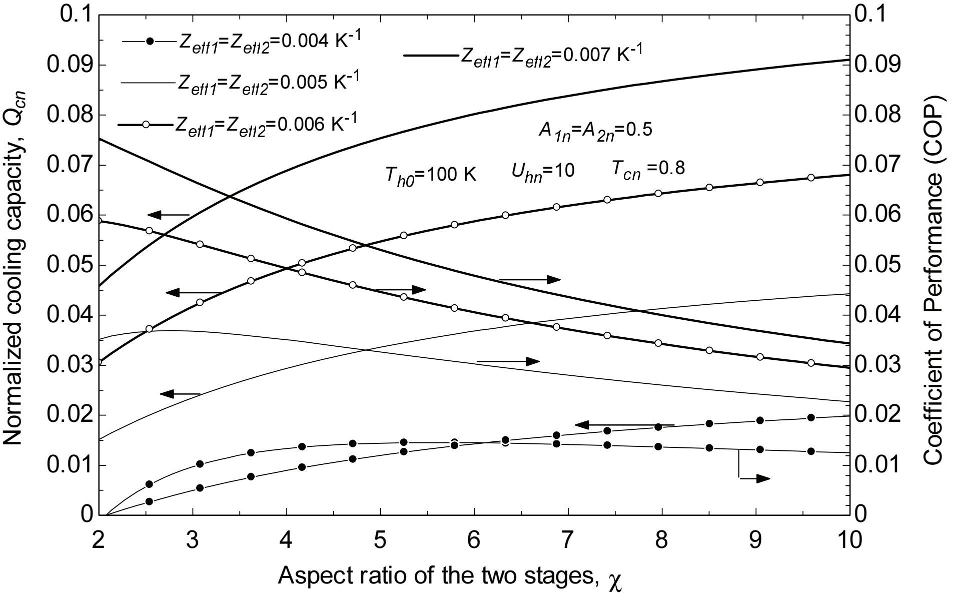

Figure 5. The normalized optimum cooling capacity and corresponding COP as a function of aspect ratio of the two stages for four values of the effective figure of merit of thermoelectric materials.

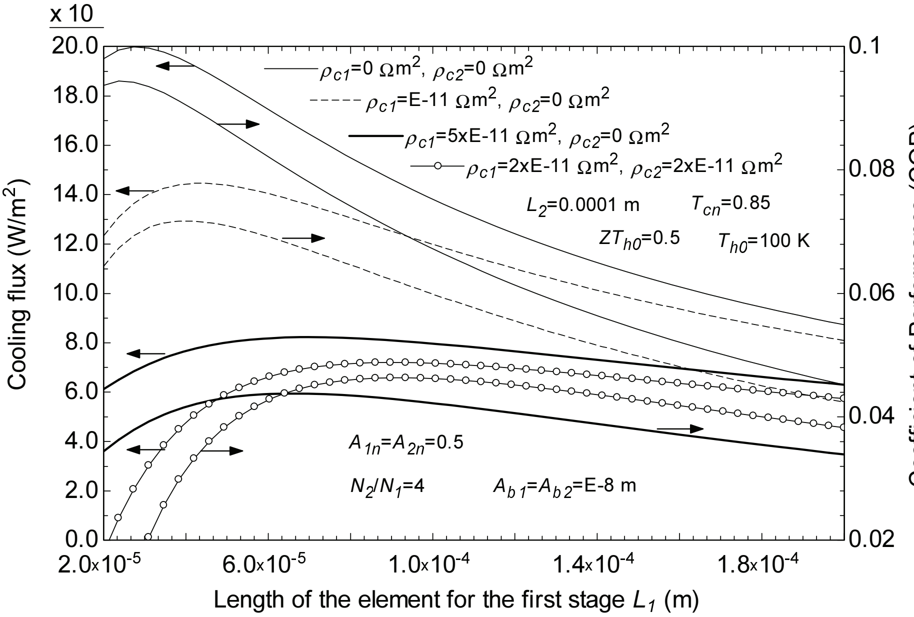

Figure 6. Optimum cooling flux and corresponding COP as a function of the length of elements in the first stage for difference values of the electrical contact resistance.

temperature of the hot reservoir. The results for other values of thermal conductance U_{hn} = 10 and U_{hn} = 2 are also given on the figure for comparison. The effect of thermal conductance on the optimum cooling capacity is significant especially for low values of thermal conductance at the hot side of refrigerator. For U_{hn} = 2 the optimum cooling capacity is negative for low values of the aspect ratio of the two stages and the no-load temperature is reached for the values approaching χ =3. Figure 3 shows the normalized optimum electric currents for the two stages given by Eqs. (3) and (10) for the same conditions of Fig. 2. The optimum current to each stage is for the condition of maximum cooling capacity of the thermoelectric refrigerator.

Figure 4 shows the normalized optimum cooling capacity and corresponding COP for the normalized aspect ratio of the two stages for different values of normalized cold end temperatures. The results are given for a moderate value of normalized thermal conductance of U_{hn} = 10 and three values of normalized cold end temperature of T_{cn}=0.85, T_{cn}=0.8 and T_{cn}=0.75. The effective figure of merit for both stages is kept constant at Z_{eff} =0.005 and other parameters are given in the figure. As expected, the effect of cold end temperature on the optimum cooling capacity and corresponding COP is significant. Figure 5 shows the effect of the figure of merit on the performance of two-stage thermoelectric refrigerator. The optimum cooling capacity and corresponding COP is given for four different values of the figure of merit for a moderate thermal conductance at the hot side of the refrigerator of U_{hn} =10. The significant effect of the figure of merit on the performance of the refrigerator is clearly seen. The optimum cooling capacity and corresponding COP increases by a factor of approximately ten when the effective figure of merit is increased from Z_{eff} = 0.004 K^{-1} to Z_{eff} = 0.007 K^{-1} when the normalized aspect ratio of the two stages is χ=6. The effect is significant for all values of χ.

Figure 6 shows the effect of variation in the length of the elements of the first stage on the cooling flux of the two-stage thermoelectric refrigerator defined by $q_c'' = Q_{c,max}/2N_1A_{b1}$. The maximum cooling power is obtained from the solution to Eqs. (8), (14), and (16) to (19). The cooling flux can be written in terms of the normalized cooling capacity given by $q_c'' = Q_{cn}T_{ho}k_1/L_1$. The variation of cooling flux with the changes in the length of the elements is complicated due to the fact that the effective figure of merit of both stages change when the effect of electric contact resistance are taken into account as shown by Eqs. (6) and (13). The results shown in Fig. (6) correspond to a fixed length of the elements in the second stage L_2 = 0.0001m, and a fixed ratio of number of elements in the second stage to the number of elements in the first stage, N_2/N_1 =4. Therefore the value of $\chi=4L_1/L_2$ for the cases studied in Fig. (6). The results in the figure are given for three values of electric contact resistance for the first stage while no contact resistance for the second stage is used. The results of calculations when the electric contact resistances for both stages are equal to 2x10^{-11} Ωm^2 are given in the figure for comparison. Other values used in the calculations are given in the figure. The value of the cooling flux given in Fig. (6) has a maximum at a particular length of element in the first stage for the cases studied. The maximum values are reduced as the electric contact resistance for the first stage is increased. The values of COP for the refrigerator corresponding to the results for the heat flux are given on the right Y-axis for comparison.

CONCLUSIONS

A thermodynamic model for the analysis of two-stage thermoelectric refrigerator operating at cryogenic temperature is developed including the most important non-dimensional parameters influencing the design and performance of the refrigerator. The model includes the effect of thermal resistance at the hot side of the refrigerator connecting it to a thermal reservoir at a cryogenic temperature. An analytical model is developed using the method of Lagrange multipliers to obtain the optimum cooling capacity of the two-stage refrigerator in terms of important non-dimensional parameters convenient for parametric studies and design analysis. An important non-dimensional parameter in design of two-stage thermoelectric refrigerator is the normalized aspect ratio of the two stages. As expected, the effective figure of merit for thermoelectric materials plays an important role in optimum cooling capacity of the two-stage refrigerator operating at the cryogenic temperature for all values of normalized aspect ratio of the two stages. In addition, the COP of the refrigerator is small at the optimum cooling capacity when the effective figure of merit of thermoelectric material is not high. It is also shown that thermal resistance at the hot side of the thermoelectric refrigerator has a significant effect on its performance when the thermal resistance is high. However, thermoelectric refrigerators can produce a high cooling flux even for moderate values of the figure of merit and the thermal resistance.

NOMENCLATURE

A_1 *and* A_2	cross sectional area for each TE element for stages 1 and 2
A_{b1} *and* A_{b2}	area of the base for each TE element for stages 1 and 2
I_1 *and* I_2	electric current for stages 1 and 2
k_1 *and* k_2	thermal conductivity for each TE element for stages 1 and 2
L_1 *and* L_2	the length of each TE element for stages 1 and 2
N_1 and N_2	number of TE elements for stages 1 and 2
Q_c	cooling capacity of the refrigerator
T_c, T_h and T_m	temperature of the cold side, hot side and mid-stage
U_h	thermal conductance at the hot side
Greek symbols	
α_1 and α_2	Seebeck coefficient for each TE element for stages 1 and 2
ρ_1 and ρ_2	electric resistivity for each TE element for stages 1 and 2
ρ_{c1} and ρ_{c2}	electric contact resistance for each TE element for stages 1 and 2

REFERENCES

1. Rowe, D.M., *Handbook of Thermoelectric: Macro to Nano,* CRC, Boca Raton, FL, (2006).
2. DiSalvo, F.J., "Thermoelectric Cooling and Power Generation," *Science*, 285 (1999), pp. 703-706.

3. Lenoir, B., Dauscher, A., Cassart, M., Ravich, Yu.I., and Scherrer, H. “Effect of antimony content on the thermoelectric figure of merit of Bi1-xSbx alloy,” *J. Phys. Chem Solids*, 59 (1998), pp. 129-134.

4. Benttien, A., Johnsen, S., Madsen, G.K.H., Iversen, B.B., Steglich, F., “Colossal Seebeck Coefficient in Strongly Correlated Semiconductor FeSb2,” *Europhysics letters*, 80 17008 (2007), pp. 1-5.

5. Hermans, P.J., “Cryogenic thermoelectric materials,” *Thermal Science and Materials Workshop*, AFRL UES Conference; Dayton, Ohio, 2011.

6. Bulman, G.E., Siivola, E.D., Wiitala, R., Venkatasubramanian, R. Acree, M., and Ritz, N. “Three-stage Thin-film Superlattice Thermoelectric Multistage Microcoolers with a Tmax of 102 K,” *Journal of Electronic Materials*, 38, (2009), pp. 1510-1515.

7. Chen, L., Li, J., Sun, F., and Wu, C., “Effect of Heat Transfer on the Performance of Two-stage Semiconductor Refrigerator,” *Journal of Applied Physics*, 98 (2005), pp. 1-6.

8. Xuan, X.C., Ng, K.C., Yap, C., and Chua, H.T., “Optimization of Two-Stage Thermoelectric Coolers with Two Design Configurations,” *Energy Conversion and Management*, 43 (2002), pp. 2041-2052.

9. Yu, J., Zhao, H., and Xie, K., “Analysis of Optimum Configuration of Two-Stage Thermoelectric modules,” *Cryogenics*, 47 (2007), pp. 90-93.

10. Yang, R., Chen, G., Snyder, G.J., and Fleurial, J., “Multistage Thermoelectric Microcoolers,” *Journal of Applied Physics*, 95 (2004), pp. 8226-8232.

11. Pettes, A.M., Hodes, M.S., and Goodson, K.F., “Optimized Thermoelectric Refrigerator in the Presence of Thermal Boundary Resistance,” *IEEE Transaction on Advanced Packaging*, 32 (2009), pp. 423-430.

12. Razani, A., Fraser, T., Dodson, C., “Performance of Thermoelectric Coolers with Boundary Resistance for Different Optimization Criteria,” *Cryocoolers 17*, ICC Press, Boulder, CO (2012), pp. 495-502.

13. Razani, A., C. Dodson, and Dodson, C., “A Thermodynamic Model for the Effect of Thermal Boundary Resistance on Multistage Thermoelectric Cryogenic Refrigerators,” *Advances in Cryogenic Engineering*, Vol. 57, Amer. Institute of Physics, Melville, NY (2012), pp. 1899-1907.